Oscillation constants for second-order nonlinear dynamic equations on time scales (Qualitative Theory on ODEs and their applications to Mathematical Modeling)

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Oscillation constants for second-order nonlinear dynamic equations on time scales

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1 Introduction

Consider second-order nonlinear dynamic equations on time scales of the form

$$x^{\Delta\Delta} + p(t)f(x) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \subset (0, \infty), \tag{1.1}$$

where a time scale $\mathbb{T}$ is assumed to be unbounded from above, x^{Δ} is the *delta-derivative* of x, $p(t)$ is an *rd-continuous* function on $[t_0, \infty)$, and $f(x)$ is a continuous function on $\mathbb{R}$ satisfying

$$xf(x) > 0 \quad \text{if } \ x \neq 0. \tag{1.2}$$

Here, for simplicity, we use the notation $I_{\mathbb{T}} = I \cap \mathbb{T}$ for the interval $I \subset \mathbb{R}$. Moreover, we use the following notation concerning time scales calculus: σ, ρ, μ, x^{σ}, $\int_a^b g(s)\,\Delta s$, $C_{\mathrm{rd}}(I)$, and $e_u(t,s)$, with the standard meaning, i.e., *forward jump operator*, *backward jump operator*, *graininess*, $x \circ \sigma$, *delta integral*, *the set of rd-continuous functions*, and *generalized exponential function*, respectively (for these definitions, see [2, 3]).

A function x is said to be a *solution* of equation (1.1) if $x \in C^2_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}})$ and x satisfies equation (1.1) for all $t \in [t_0, \infty)_{\mathbb{T}}$. Throughout this paper, we assume that all solutions of equation (1.1) exist in the future. Then we can discuss the oscillatory behavior of solutions of equation (1.1) as $t \to \infty$. Here a solution $x(t)$ of equation (1.1) is said to be *nonoscillatory* if it is either eventually positive or eventually negative, otherwise it is said to be *oscillatory*.

Equation (1.1) naturally includes the Euler-Cauchy dynamic equation

$$y^{\Delta\Delta} + \frac{\lambda}{t\sigma(t)}y = 0, \quad t \in [t_0, \infty)_{\mathbb{T}} \tag{1.3}$$

as a special case, where $\lambda > 0$. It is known that equation (1.3) has the general solution

$$y(t) = \begin{cases} K_1 e_{z/t}(t, t_0) + K_2 e_{(1-z)/t}(t, t_0) & \text{if } \ \lambda \neq 1/4, \\ e_{1/(2t)}(t, t_0)\left\{K_3 + K_4 \displaystyle\int_{t_0}^{t} \frac{2}{s + \sigma(s)}\Delta s\right\} & \text{if } \ \lambda = 1/4, \end{cases}$$

where K_1, K_2, K_3, K_4 are arbitrary constants and z is the root of the characteristic equation

$$z^2 - z + \lambda = 0.$$

Hence, we have the following result (for the proof, see [6]).

Proposition 1.1. *Equation* (1.3) *can be classified into two types as follows:*

(i) if $\lambda > 1/4$*, then all nontrivial solutions of equation* (1.3) *are oscillatory;*

(ii) if $0 < \lambda \le 1/4$*, then all nontrivial solutions of equation* (1.3) *are nonoscillatory.*

Thus, we see that the constant $1/4$ is the critical value for the oscillation of equation (1.3). Such a value is generally called an *oscillation constant*.

When $\mathbb{T} = \mathbb{R}$, equation (1.3) becomes the linear differential equation

$$y'' + \frac{\lambda}{t^2} y = 0, \quad t \in [t_0, \infty). \tag{1.4}$$

It is known that the oscillation constant for equation (1.4) plays an important role in proving (non)oscillation criteria for equation (1.1) with $\mathbb{T} = \mathbb{R}$, i.e., the nonlinear differential equation

$$x'' + p(t) f(x) = 0, \qquad t \in [t_0, \infty). \tag{1.5}$$

For example, these results can be found in [4, 5, 7, 8, 9] and the references cited therein. Especially, Sugie and Kita [8] gave (non)oscillation criteria for equation (1.5) which can be applied even to the critical case $f(x)/x \searrow 1/4$ as $|x| \to \infty$.

On the other hand, the author [10, 11] discussed the oscillation problem for equation (1.1) with $\mathbb{T} = \mathbb{N}$, i.e., the nonlinear difference equation

$$\Delta^2 x(t) + p(t) f(x(t)) = 0, \quad t \in [t_0, \infty)_{\mathbb{N}},$$

where Δ is the forward difference operator, and gave (non)oscillation criteria which can be regarded as counterparts of the results of Sugie and Kita [8].

In this paper, we intend to unify these results. For this purpose, we give a pair of an oscillation theorem and a nonoscillation theorem for equation (1.1). Our main results are stated as follows.

Theorem 1.1. *Assume* (1.2)*. Suppose that* $p(t)$ *satisfies*

$$t\sigma(t)p(t) \ge 1 \tag{1.6}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$ *sufficiently large, and that there exists a* $\lambda \in \mathbb{R}$ *with* $\lambda > 1/4$ *such that*

$$\frac{f(x)}{x} \ge \lambda \tag{1.7}$$

for $|x|$ *sufficiently large. Then all nontrivial solutions of equation* (1.1) *are oscillatory.*

Theorem 1.2. *Assume* (1.2)*. Suppose that* $p(t)$ *satisfies*

$$t\sigma(t)p(t) \le 1 \tag{1.8}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large, and that

$$\frac{f(x)}{x} \le \frac{1}{4} \tag{1.9}$$

for $x > 0$ or $x < 0$, $|x|$ sufficiently large. Then equation (1.1) *has a nonoscillatory solution.*

2 Proof of oscillation criteria

To begin with we prepare some lemmas which are useful for proving Theorem 1.1.

Lemma 2.1. *Let $0 < a \in \mathbb{T}$. Then, $\int_a^\infty \Delta t/t = \infty$.*

Proof. To complete the proof, it suffices to show that,

$$\int_a^t \frac{ds}{s} \le \int_a^t \frac{\Delta s}{s}$$

holds for any unbounded time scale $\mathbb{T}$, $0 < a \in \mathbb{T}$, and $a < t \in \mathbb{T}$, because $\int_a^t ds/s = \log(t/a) \to \infty$ as $t \to \infty$.

Suppose that there exist an unbounded time scale $\mathbb{T}_0$, $\varepsilon_0 > 0$, $0 < a_0 \in \mathbb{T}_0$, and $a_0 < t_0 \in \mathbb{T}_0$ such that

$$\int_{a_0}^{t_0} \frac{ds}{s} > \int_{a_0}^{t_0} \frac{\Delta s}{s} + \varepsilon_0.$$

Then, in view of the definition of the delta Riemann type integral (see [3, Chapter 5]), there exists a discrete time scale $\tilde{\mathbb{T}}$ containing a_0 and t_0 such that

$$\left| \int_{a_0}^{t_0} \frac{\tilde{\Delta} s}{s} - \int_{a_0}^{t_0} \frac{\Delta s}{s} \right| < \frac{\varepsilon_0}{2},$$

where $\int_{a_0}^{t_0} \tilde{\Delta} s/s$ is the delta integral with respect to $\tilde{\mathbb{T}}$. Since

$$\int_{a_0}^{t_0} \frac{\tilde{\Delta} s}{s} = \sum_{s \in [a_0, t_0)_{\tilde{\mathbb{T}}}} \frac{\tilde{\mu}(s)}{s},$$

it is clear that $\int_{a_0}^{t_0} \frac{ds}{s} \le \int_{a_0}^{t_0} \frac{\tilde{\Delta} s}{s}$, where $\tilde{\mu}$ stands for the graininess associated to $\tilde{\mathbb{T}}$. Hence, we have

$$\int_{a_0}^{t_0} \frac{\Delta s}{s} + \varepsilon_0 < \int_{a_0}^{t_0} \frac{ds}{s} \le \int_{a_0}^{t_0} \frac{\tilde{\Delta} s}{s} < \int_{a_0}^{t_0} \frac{\Delta s}{s} + \frac{\varepsilon_0}{2},$$

which is a contradiction. □

Lemma 2.2. *Assume* (1.2) *and* (1.6)*. Suppose that equation* (1.1) *has a positive solution $x(t)$. Then the solution $x(t)$ is increasing for $t \in [t_0, \infty)_{\mathbb{T}}$ sufficiently large and it tends to ∞ as $t \to \infty$.*

Proof. By the assumption, there exists $t_1 \in \mathbb{T}$ such that $x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Hence, in view of (1.2) and (1.6), we have

$$x^{\Delta\Delta}(t) = -p(t)f(x(t)) < 0 \tag{2.1}$$

for $t \in [t_1, \infty)_{\mathbb{T}}$.

We first show that $x^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. By way of contradiction, we suppose that there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x^{\Delta}(t_2) \leq 0$. Then, by (2.1), we have $x^{\Delta}(t) < x^{\Delta}(t_2) \leq 0$ for $t \in (t_2, \infty)_{\mathbb{T}}$. Therefore, we can find $t_3 \in (t_2, \infty)_{\mathbb{T}}$ such that $x^{\Delta}(t_3) < 0$. Integrating both sides of (2.1), we get $x^{\Delta}(t) \leq x^{\Delta}(t_3) < 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$. Hence, we obtain $x(t) \leq x^{\Delta}(t_3)(t - t_3) + x(t_3) \to -\infty$ as $t \to \infty$. This contradicts the assumption that $x(t)$ is positive for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus $x(t)$ is increasing for $t \in [t_1, \infty)_{\mathbb{T}}$.

We next suppose that $x(t)$ is bounded from above. Then there exists $K > 0$ such that $\lim_{t\to\infty} x(t) = K$. Since $f(x)$ is continuous on $\mathbb{R}$, we have $\lim_{t\to\infty} f(x(t)) = f(K)$, and therefore, there exists $t_4 \in [t_1, \infty)_{\mathbb{T}}$ such that $0 < f(K)/2 < f(x(t))$ for $t \in [t_4, \infty)_{\mathbb{T}}$. Integrating both sides of (2.1) from t to $2t$ and using (1.6), we have

$$x^{\Delta}(t) = x^{\Delta}(2t) + \int_t^{2t} p(s)f(x(s))\Delta s > \frac{f(K)}{2}\int_t^{2t} \frac{\Delta s}{s\sigma(s)} = \frac{f(K)}{4t}$$

for $t \in [t_4, \infty)_{\mathbb{T}}$, and therefore, we obtain

$$x(t) \geq x(t_4) + \frac{f(K)}{4}\int_{t_4}^{t} \frac{\Delta s}{s} \to \infty$$

as $t \to \infty$ by Lemma 2.1. This contradicts the assumption that $x(t)$ is bounded from above. Thus we conclude that $\lim_{t\to\infty} x(t) = \infty$. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $t_0 \in \mathbb{T}$ be a large number satisfying (1.6) for $t \in [t_0, \infty)_{\mathbb{T}}$ and let $R > 0$ be a large number such that (1.7) is satisfied for $|x| \geq R$.

The proof is by contradiction. Suppose that equation (1.1) has a nonoscillatory solution $x(t)$. Then, without loss of generality, we may assume that $x(t)$ is eventually positive. By Lemma 2.2, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) \geq R$ and $x^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$.

We define

$$w(t) = \frac{x^{\Delta}(t)}{x(t)}.$$

Then $w(t)$ is positive and satisfies

$$\begin{aligned} w^{\Delta}(t) =& \frac{x^{\Delta\Delta}(t)x(t) - (x^{\Delta}(t))^2}{x(t)x^{\sigma}(t)} = \frac{x^{\Delta\Delta}(t)x(t) + (\mu(t)x^{\Delta\Delta}(t) - (x^{\Delta}(t))^{\sigma})x^{\Delta}(t)}{x(t)x^{\sigma}(t)} \\ =& \frac{(x(t) + \mu(t)x^{\Delta}(t))x^{\Delta\Delta}(t) - (x^{\Delta}(t))^{\sigma}x^{\Delta}(t)}{x(t)x^{\sigma}(t)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{\Delta\Delta}(t)}{x(t)} - w(t)w^\sigma(t) = -\frac{p(t)f(x(t))}{x(t)} - w(t)w^\sigma(t) \\
&\leq -\frac{\lambda}{t\sigma(t)} - w(t)w^\sigma(t) \qquad (2.2)
\end{aligned}
$$

for $t \in [t_1, \infty)_{\mathbb{T}}$ by (1.6) and (1.7).

We will show that $\lim_{t\to\infty} w(t) = 0$. Since $w^\Delta(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$, we see that $w(t) \searrow \alpha \in [0, \infty)$ as $t \to \infty$. Assume that $\alpha > 0$. Then, from (2.2), we have

$$
\left(-\frac{1}{w(t)}\right)^\Delta = \frac{w^\Delta(t)}{w(t)w^\sigma(t)} \leq -\frac{\lambda}{t\sigma(t)w(t)w^\sigma(t)} - 1 < -1.
$$

Hence, using $w(t) \geq \alpha > 0$, we get

$$
\frac{1}{w(t_1)} - \frac{1}{\alpha} \leq \frac{1}{w(t_1)} - \frac{1}{w(t)} \leq -t + t_1 \to \infty
$$

as $t \to \infty$. This is a contradiction. Hence, we obtain $\alpha = 0$.

Integrating both sides of (2.2) from t to s, we have

$$
\int_t^s \left(\frac{\lambda}{\tau\sigma(\tau)} + w(\tau)w^\sigma(\tau)\right) \Delta\tau \leq w(t) - w(s) \leq w(t).
$$

Letting $s \to \infty$, we get

$$
w(t) \geq \frac{\lambda}{t} + \int_t^\infty w(s)w^\sigma(s)\,\Delta s \qquad (2.3)
$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. Define the sequence $\{u_k\}$ as follows:

$$
u_0(t) = \frac{\lambda}{t}, \quad u_k(t) = u_0(t) + \int_t^\infty u_{k-1}(s)u_{k-1}^\sigma(s)\,\Delta s \qquad (2.4)
$$

for $t \in [t_1, \infty)_{\mathbb{T}}$, $k = 0, 1, 2, \ldots$. Note that the sequence $\{u_k\}$ is well defined. In fact, we have $u_0(t) \leq w(t)$, and therefore, from (2.3), we obtain

$$
u_1(t) = u_0(t) + \int_t^\infty u_0(s)u_0^\sigma(s)\,\Delta s \leq u_0(t) + \int_t^\infty w(s)w^\sigma(s)\,\Delta s \leq w(t),
$$

for $t \in [t_1, \infty)_{\mathbb{T}}$. By induction, we obtain $0 < u_k(t) \leq w(t)$ and $u_k(t) \leq u_{k+1}(t)$ for $t \in [t_1, \infty)_{\mathbb{T}}$, $k = 0, 1, 2, \ldots$. Hence, $\{u_k\}$ is monotone and bounded from above, thus there exists the limit $\lim_{k\to\infty} u_k(t) = u(t)$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Applying the Lebesgue monotone convergence theorem on time scales (see e.g., [1]) to (2.4), we find that u satisfies the equation

$$
u(t) = \frac{\lambda}{t} + \int_t^\infty u(s)u^\sigma(s)\,\Delta s.
$$

Differentiating both sides of the equality, we have

$$
u^\Delta(t) = -\frac{\lambda}{t\sigma(t)} - u(t)u^\sigma(t).
$$

Define $y(t) = e_u(t, t_1) > 0$. Then, since $u(t) = y^\Delta(t)/y(t)$, we have

$$u^\Delta(t) = \frac{y^{\Delta\Delta}(t)}{y(t)} - u(t)u^\sigma(t),$$

and therefore, $y(t)$ is a positive solution of equation (1.3). Hence, we see that equation (1.3) has a nonoscillatory solution $y(t)$. However, since $\lambda > 1/4$, all nontrivial solutions of equation (1.3) are oscillatory by Proposition 1.1. This is a contradiction. The proof is now complete. □

3 Proof of nonoscillation criteria

To prove Theorem 1.2, we require some lemmas.

Lemma 3.1. *Let* $0 < a \in \mathbb{T}$. *Then* $\int_a^\infty \Delta t/\sigma(t) = \infty$.

Proof. Let $I = \{t \in \mathbb{T} : \mu(t) \geq t\}$. Suppose that I is bounded from above. Then there exists $a \leq t_1 \in \mathbb{T}$ such that $\mu(t) < t$ for $t \in [t_1, \infty)_\mathbb{T}$. Hence, we have

$$\int_a^\infty \frac{\Delta t}{\sigma(t)} \geq \int_{t_1}^\infty \frac{\Delta t}{\sigma(t)} = \int_{t_1}^\infty \frac{\Delta t}{t + \mu(t)} \geq \int_{t_1}^\infty \frac{\Delta t}{2t} = \infty$$

by Lemma 2.1. On the contrary, if I is unbounded from above, we obtain

$$\int_a^\infty \frac{\Delta t}{\sigma(t)} \geq \sum_{t \in I} \int_t^{\sigma(t)} \frac{\Delta s}{\sigma(s)} = \sum_{t \in I} \frac{\mu(t)}{\sigma(t)} = \sum_{t \in I} \frac{\mu(t)}{t + \mu(t)} \geq \sum_{t \in I} \frac{1}{2} = \infty.$$

The proof is now complete. □

We next present a comparison lemma.

Lemma 3.2. *Assume that* $g(t, x)$ *is a continuous function such that*

$$x \mapsto x + \mu(t)g(t, x) \text{ is nondecreasing for each fixed } t. \tag{3.1}$$

If φ *and* ψ *satisfy* $\psi(a) \geq \varphi(a)$, $\varphi^\Delta = g(t, \varphi(t))$, $\psi^\Delta(t) > g(t, \psi(t))$ *for* $t \in [a, b]_\mathbb{T}^\kappa$ *then*

$$\psi(t) \geq \varphi(t) \tag{3.2}$$

for $t \in [a, b]_\mathbb{T}$.

Proof. We use the following abbreviations: ld for left-dense, rd for right-dense, ls for left-scattered, and rs for right-scattered. Let

$$A_\mathbb{T} = \{t_n \in [a, b]_\mathbb{T} : t_n \text{ is ld and rs or } t_n = a \text{ or } t_n = b, n \in \mathbb{N}\}$$

with $t_n < t_{n+1}$. Then $t_1 = a$ and $A_{\mathbb{T}}$ is at most countable. Indeed, every (real) interval (t_n, t_{n+1}) can be represented by a rational number which is contained in it, and these intervals are pairwise disjoint.

We will show that

$$\psi(t) > \varphi(t) \text{ for } t \in (t_n, t_{n+1})_{\mathbb{T}}, \text{ and } \psi(t_n) \geq \varphi(t_n) \text{ for all } n, \tag{3.3}$$

which then implies (3.2).

Assume that a is rd. We claim that there is a right neighborhood U of a (more precisely, $U = (a, a + \varepsilon) \cap \mathbb{T}$ with some $\varepsilon > 0$) such that $\psi(t) > \varphi(t)$ for $t \in U$. Indeed, if $\psi(a) > \varphi(a)$, then the existence of U clearly follows from the continuity of ψ and φ; note that ψ, φ are continuous thanks to their Δ-differentiability, see [2, Theorem 1.16]. If $\psi(a) = \varphi(a)$, then $\psi^{\Delta}(a) > g(a, \psi(a)) = g(a, \varphi(a)) = \varphi^{\Delta}(a)$. Hence, $(\varphi - \psi)^{\Delta}(a) < 0$, which implies $(\varphi - \psi)(t) < 0$ for $t \in U$ by [3, Theorem 1.9].

If a is rs, then $\psi(\sigma(a)) > \psi(a) + \mu(a)g(a, \psi(a)) \geq \varphi(a) + \mu(a)g(a, \varphi(a)) = \varphi(\sigma(a))$, and so $\psi(\sigma(a)) > \varphi(\sigma(a))$.

Suppose that there exists $c \in (t_1, t_2) = (a, t_2)$ such that $\psi(t) > \varphi(t)$ for $t \in (a, c)_{\mathbb{T}}$ and $\psi(c) \leq \varphi(c)$. Then, in view of $\psi(t) > \varphi(t)$ for $t \in (a, c)_{\mathbb{T}}$ and the continuity of ψ, φ, we see that the strict inequality in $\psi(c) \leq \varphi(c)$ cannot happen when c is ld. Assume that c is ld. Then $\psi(c) = \varphi(c)$, and so

$$\psi^{\Delta}(c) > g(c, \psi(c)) = g(c, \varphi(c)) = \varphi^{\Delta}(c). \tag{3.4}$$

On the other hand, since $c \notin A_{\mathbb{T}}$, we see that c is ld and rd, and so

$$\psi^{\Delta}(c) \leq \varphi^{\Delta}(c), \tag{3.5}$$

which contradicts (3.4). To see (3.5), note that

$$\lim_{t \to c-} \frac{\psi(c) - \psi(t)}{c - t} \leq \lim_{t \to c-} \frac{\varphi(c) - \varphi(t)}{c - t},$$

in view of $\psi > \varphi$ on $(a, c)_{\mathbb{T}}$, and

$$\varphi^{\Delta}(c) = \lim_{t \to c} \frac{\varphi(c) - \varphi(t)}{c - t}, \quad \psi^{\Delta}(c) = \lim_{t \to c} \frac{\psi(c) - \psi(t)}{c - t},$$

which follows from [2, Theorem 1.16]. Assume now that c is ls. Then we see that $\rho(c) < c$, and therefore, $\sigma(\rho(c)) = c$. Hence, from $\psi(\rho(c)) > \varphi(\rho(c))$ and

$$\begin{aligned} \varphi(c) &= \varphi(\sigma(\rho(c))) = \varphi(\rho(c)) + \mu(\rho(c))g(\rho(c), \varphi(\rho(c))), \\ \psi(c) &= \psi(\sigma(\rho(c))) > \psi(\rho(c)) + \mu(\rho(c))g(\rho(c), \psi(\rho(c))), \end{aligned}$$

we get $\psi(c) > \varphi(c)$, again contradiction with $\psi(c) \leq \varphi(c)$. Therefore, $\psi(t) > \varphi(t)$ for $t \in (a, t_2)_{\mathbb{T}}$. This implies $\psi(t_2) \geq \varphi(t_2)$ since t_2 is ld and ψ, φ are continuous.

Similarly we prove that $\psi(t_n) \geq \varphi(t_n)$ implies $\psi(t) > \varphi(t)$ for $t \in (t_n, t_{n+1})_{\mathbb{T}}$ and $\psi(t_{n+1}) \geq \varphi(t_{n+1})$ for all n. Thus, by induction, (3.3) follows. $\square$

Let $x(t)$ be a solution of equation (1.1) and put $y(t) = tx^{\Delta}(t) - x(t)$. Then we can transform equation (1.1) into the system

$$tx^{\Delta} = y + x, \qquad ty^{\Delta} = -t\sigma(t)p(t)f(x). \tag{3.6}$$

For simplicity, put

$$D_1 = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y \geq -x\}, \quad D_2 = \{(x,y) \in \mathbb{R}^2 : x > 0,\ y < -x\}.$$

Then we have the following lemma.

Lemma 3.3. *Let* $(x(t), y(t))$ *be a solution of* (3.6) *which corresponds to a nontrivial oscillatory solution of equation* (1.1). *If* $t_0 \in \mathbb{T}$ *satisfies* $(x(t_0), y(t_0)) \in D_1$, *then there exists* $t_1 \in \mathbb{T}$ *such that* $(x(t), y(t)) \in D_1$ *on* $[t_0, t_1)_{\mathbb{T}}$ *and* $(x(t_1), y(t_1)) \in D_2$.

Proof. Since $x(t)$ is oscillatory, there exists $\tilde{t} \in \mathbb{T}$ such that $x(t) > 0$ on $[t_0, \tilde{t})_{\mathbb{T}}$ and $x(\tilde{t}) \leq 0$.

Let $\tilde{t}$ be left-scattered. Then we have $x(\rho(\tilde{t})) > 0$, and therefore, $(x(\rho(\tilde{t})), y(\rho(\tilde{t})))$ is either in D_1 or D_2. Suppose that $(x(\rho(\tilde{t})), y(\rho(\tilde{t}))) \in D_1$. Then we see that

$$\rho(\tilde{t})x^{\Delta}(\rho(\tilde{t})) = y(\rho(\tilde{t})) + x(\rho(\tilde{t})) \geq 0.$$

Hence, we have $x(\tilde{t}) = x(\sigma(\rho(\tilde{t}))) \geq x(\rho(\tilde{t})) > 0$, which is a contradiction. Thus we obtain $(x(\rho(\tilde{t})), y(\rho(\tilde{t}))) \in D_2$, that is, there exists $t_1 \in (t_0, \rho(\tilde{t})]_{\mathbb{T}}$ such that $(x(t), y(t)) \in D_1$ on $[t_0, t_1)_{\mathbb{T}}$ and $(x(t_1), y(t_1)) \in D_2$.

Let $\tilde{t}$ be left-dense. Then there is a left neighborhood $U \subset [t_0, \tilde{t})_{\mathbb{T}}$ of $\tilde{t}$ such that $x(t) > 0$ for $t \in U$. Suppose that $(x(t), y(t)) \in D_1$ for $t \in U$. Since $tx^{\Delta}(t) = y(t) + x(t) \geq 0$, in view of [3, Corollary 1.20] and the continuity of x, we have $x(\tilde{t}) = \lim_{t \to \tilde{t}-} x(t) > 0$, which is a contradiction. Thus there exists $t_2 \in U$ such that $(x(t_2), y(t_2)) \in D_2$, that is, the assertion of this lemma holds. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. We prove only the case that condition (1.9) is satisfied for $x > 0$ sufficiently large, because the other case can be proved in the same manner. Let $R > 0$ be a number such that (1.9) is satisfied for $x \geq R$. Moreover, we suppose that there exists $T \in \mathbb{T}$ such that $p(t)$ satisfies (1.8) for $t \in [T, \infty)_{\mathbb{T}}$.

The proof is by contradiction. Suppose that all nontrivial solutions of equation (1.1) are oscillatory. Let $(x(t), y(t))$ be the solution of system (3.6) satisfying the initial condition

$$(x(t_0), y(t_0)) = \left(R, \left(-\frac{1}{2} + \frac{1}{2l(t_0)}\right) R\right) \in D_1,$$

where $t_0 \in [T, \infty)_{\mathbb{T}}$ and the function $l(t)$ is positive and satisfies $l^{\Delta}(t) = 2/(t + \sigma(t))$ on $[t_0, \infty)_{\mathbb{T}}$. Note that $l(t) \to \infty$ as $t \to \infty$ because of Lemma 3.1. By Lemma 3.3, there exists $t_1 \in \mathbb{T}$ such that

$$(x(t), y(t)) \in D_1 \quad \text{on} \quad [t_0, t_1)_{\mathbb{T}} \quad \text{and} \quad (x(t_1), y(t_1)) \in D_2.$$

Note that $x(t) \geq R$ on $[t_0, t_1]_{\mathbb{T}}^{\kappa}$ because $tx^{\Delta}(t) = y(t) + x(t) > 0$ on $[t_0, t_1)_{\mathbb{T}}$.

Define $\psi(t) = y(t)/x(t)$. Then $\psi(t)$ satisfies

$$\psi(t_0) = \frac{y(t_0)}{x(t_0)} = -\frac{1}{2} + \frac{1}{2l(t_0)}, \quad \text{and} \quad \psi(t_1) = \frac{y(t_1)}{x(t_1)} < -1. \tag{3.7}$$

Moreover, using (1.8) and (1.9), we easily see that $\psi(t)$ satisfies

$$\begin{aligned}
\psi^{\Delta}(t) =& \frac{y^{\Delta}(t)x(t) - y(t)x^{\Delta}(t)}{x(t)x^{\sigma}(t)} = \frac{x(t)}{tx^{\sigma}(t)} \left(\frac{ty^{\Delta}(t)}{x(t)} - \frac{y(t)tx^{\Delta}(t)}{x^2(t)} \right) \\
=& \frac{x(t)}{tx^{\sigma}(t)} \left(-\frac{t\sigma(t)p(t)f(x(t))}{x(t)} - \frac{y(t)(y(t) + x(t))}{x^2(t)} \right) \\
\geq& \frac{x(t)}{tx^{\sigma}(t)} \left\{ -\frac{1}{4} - \left(\frac{y(t)}{x(t)} \right)^2 - \frac{y(t)}{x(t)} \right\} = -\frac{x(t)}{tx^{\sigma}(t)} \left(\psi^2(t) + \psi(t) + \frac{1}{4} \right) \\
=& -\frac{1}{\mu(t)\psi(t) + \sigma(t)} \left(\psi(t) + \frac{1}{2} \right)^2 \\
>& -\frac{1}{\mu(t)\psi(t) + \sigma(t)} \left\{ \left(\psi(t) + \frac{1}{2} \right)^2 + \frac{1}{4l(t)l^{\sigma}(t)} \right\}
\end{aligned}$$

Note that

$$\begin{aligned}
\mu(t)\psi(t) + \sigma(t) =& \mu(t)(\psi(t) + 1) + t = \mu(t) \left(\frac{tx^{\Delta}(t) - x(t)}{x(t)} + 1 \right) + t \\
=& \mu(t)\frac{tx^{\Delta}(t)}{x(t)} + t = \frac{t(\mu(t)x^{\Delta}(t) + x(t))}{x(t)} = \frac{tx^{\sigma}(t)}{x(t)} > 0
\end{aligned}$$

for $t \in [t_0, t_1]_{\mathbb{T}}^{\kappa}$.

We compare the function $\psi(t)$ with the function

$$\varphi(t) = -\frac{1}{2} + \frac{1}{2l(t)}. \tag{3.8}$$

Note that φ is a solution of the equation

$$\varphi^{\Delta}(t) = -\frac{1}{\mu(t)\varphi(t) + \sigma(t)} \left\{ \left(\varphi(t) + \frac{1}{2} \right)^2 + \frac{1}{4l(t)l^{\sigma}(t)} \right\}.$$

Indeed, by a direct computation, we have

$$
\begin{aligned}
&\frac{1}{\mu(t)\varphi(t)+\sigma(t)}\left\{\left(\varphi(t)+\frac{1}{2}\right)^2+\frac{1}{4l(t)l^\sigma(t)}\right\}\\
=&\left\{\mu(t)\left(-\frac{1}{2}+\frac{1}{2l(t)}\right)+\sigma(t)\right\}^{-1}\left\{\frac{1}{4l^2(t)}+\frac{1}{4l(t)l^\sigma(t)}\right\}\\
=&\left(\frac{\mu(t)}{2l(t)}+\frac{2\sigma(t)-\mu(t)}{2}\right)^{-1}\left(\frac{l^\sigma(t)}{2l(t)}+\frac{1}{2}\right)\frac{1}{2l(t)l^\sigma(t)}\\
=&\left(\frac{\mu(t)}{2l(t)}+\frac{t+\sigma(t)}{2}\right)^{-1}\left(\frac{l(t)+\mu(t)l^\Delta(t)}{2l(t)}+\frac{1}{2}\right)\frac{1}{2l(t)l^\sigma(t)}\\
=&\left(\frac{\mu(t)}{2l(t)}+\frac{1}{l^\Delta(t)}\right)^{-1}\left(\frac{\mu(t)l^\Delta(t)}{2l(t)}+1\right)\frac{1}{2l(t)l^\sigma(t)}=\frac{l^\Delta(t)}{2l(t)l^\sigma(t)}=-\varphi^\Delta(t).
\end{aligned}
$$

Since, with x such that $\mu(t)x+\sigma(t)\neq 0$,

$$
\begin{aligned}
&\frac{d}{dx}\left[x-\frac{\mu(t)}{\mu(t)x+\sigma(t)}\left\{\left(x+\frac{1}{2}\right)^2+\frac{1}{4l(t)l^\sigma(t)}\right\}\right]\\
=&1+\frac{\mu^2(t)}{(\mu(t)x+\sigma(t))^2}\left\{\left(x+\frac{1}{2}\right)^2+\frac{1}{4l(t)l^\sigma(t)}\right\}-\frac{2\mu(t)}{\mu(t)x+\sigma(t)}\left(x+\frac{1}{2}\right)\\
=&\left\{1-\frac{\mu(t)}{\mu(t)x+\sigma(t)}\left(x+\frac{1}{2}\right)\right\}^2+\frac{\mu^2(t)}{4l(t)l^\sigma(t)(\mu(t)x+\sigma(t))^2}\geq 0,
\end{aligned}
$$

using Lemma 3.2, we have $\varphi(t)\leq\psi(t)$ on $[t_0,t_1]_{\mathbb{T}}$ because $\psi(t_0)=\varphi(t_0)$. Hence, together with (3.7) and (3.8), we have

$$-\frac{1}{2}<\varphi(t_1)\leq\psi(t_1)<-1,$$

which is a contradiction. This completes the proof. □

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